• We define the **divergence** of \( \mathbf{F} = (F_1, F_2, F_3) \) to be:

\[
\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \nabla f
\]

• **Divergence Theorem**: If \( \mathcal{W} \) is a region in \( \mathbb{R}^3 \) whose boundary \( \partial \mathcal{W} \) is a surface, oriented by outward pointing normal vectors, then

\[
\text{Corollary: If } \int_{\partial \mathcal{W}} \mathbf{F} \cdot d\mathbf{S} = 0, \text{ then } \mathbf{F} \text{ has zero flux through the boundary } \partial \mathcal{W} \text{ of any } \mathcal{W} \text{ contained in the domain of } \mathbf{F}.
\]

• Basic operations on functions and vector fields:

\[
\begin{array}{cccc}
\text{function} & \nabla & \text{vector field} & \text{curl} & \text{vector field} & \text{div} & \text{function} \\
\text{f} & \mathbf{F} & \mathbf{G} & g
\end{array}
\]

• The result of two consecutive operations is zero:

\[
\nabla \cdot \mathbf{G} = 0, \quad \text{div}(\mathbf{F}) = 0
\]

• The “converses” of those two statements hold *over simply-connected domains*:

  - If \( \text{curl}(\mathbf{F}) = 0 \), then there is a scalar function \( f \) so that \( \mathbf{F} = \nabla f \).
  - If \( \text{div}(\mathbf{G}) = 0 \), then there is a vector field \( \mathbf{F} \) so that \( \mathbf{G} = \nabla \times \mathbf{F} \).
1. Let $S$ be the closed cylinder $x^2 + y^2 = 4$ with top $z = 4$ and bottom $z = 0$.
   
   (a) Calculate the flux of $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ out of $S$ directly.
   
   (b) Calculate the same flux using the Divergence Theorem.

2. Let $S$ be the (non-closed) hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$ oriented with the upward-pointing normal. Use the Divergence Theorem to calculate the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F} = \langle x^3 + 2xy^2 - yz, 3xz^2, y^2z + z^3 \rangle$.

3. Let $\mathbf{F} = \langle x, y, z \rangle$. Prove that if $W = \mathbb{R}^3$ is a region with smooth boundary $S$, then
   
   $$\text{volume}(W) = \frac{1}{3} \iint_S \mathbf{F} \cdot d\mathbf{S}.$$  

4. Let $W$ be the region between the sphere of radius 3 and the sphere of radius 2, both centered at the origin. Calculate the flux of $\mathbf{F} = xi$ through the boundary $S = \partial W$.

5. Find and prove a “product rule” expressing $\text{div}(f\mathbf{F})$ in terms of $\text{div}(\mathbf{F})$ and $\nabla f$. 

We define the **divergence** of \( \mathbf{F} = (F_1, F_2, F_3) \) to be:

\[
\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}
\]

**Divergence Theorem**: If \( W \) is a region in \( \mathbb{R}^3 \) whose boundary \( \partial W \) is a surface, oriented by outward pointing normal vectors, then

\[
\int \int_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \int \int \int_W \text{div}(\mathbf{F}) \, dV
\]

**Corollary**: If \( \text{div}(\mathbf{F}) = 0 \) then \( \mathbf{F} \) has zero flux through the boundary \( \partial W \) of any \( W \) contained in the domain of \( \mathbf{F} \).

**Basic operations on functions and vector fields**: 

<table>
<thead>
<tr>
<th>Function</th>
<th>( \nabla )</th>
<th>Vector Field</th>
<th>( \text{curl} )</th>
<th>Vector Field</th>
<th>( \text{div} )</th>
<th>Function</th>
</tr>
</thead>
</table>

The result of two consecutive operations is zero:

\[
\text{curl}(\nabla(f)) = 0, \quad \text{div}(\text{curl}(\mathbf{F})) = 0
\]

The “converses” of those two statements hold *over simply-connected domains*:

- If \( \text{curl}(\mathbf{F}) = 0 \), then there is a scalar function \( f \) so that \( \nabla f = \mathbf{F} \).
- If \( \text{div}(\mathbf{G}) = 0 \), then there is a vector field \( \mathbf{F} \) so that \( \text{curl}(\mathbf{F}) = \mathbf{G} \).
1. Let $S$ be the closed cylinder $x^2 + y^2 = 4$ with top $z = 4$ and bottom $z = 0$.
   
   (a) Calculate the flux of $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ out of $S$ directly.
   
   (b) Calculate the same flux using the Divergence Theorem.

   **Answer:** $48\pi$.

2. Let $S$ be the (non-closed) hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$ oriented with the upward-pointing normal. Use the Divergence Theorem to calculate the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F} = \langle x^3 + 2xy^2 - yz, 3xz^2, y^2z + z^3 \rangle$.

   **Answer:** $\frac{2 \cdot 3^6}{5}\pi$

3. Let $\mathbf{F} = \langle x, y, z \rangle$. Prove that if $W = \mathbb{R}^3$ is a region with smooth boundary $S$, then

   $$\text{volume}(W) = \frac{1}{3} \iint_S \mathbf{F} \cdot d\mathbf{S}.$$ 

   **Solution:** Apply the Divergence Theorem, noticing that $\text{div}(\mathbf{F}) = 3$.

4. Let $W$ be the region between the sphere of radius 3 and the sphere of radius 2, both centered at the origin. Calculate the flux of $\mathbf{F} = x\mathbf{i}$ through the boundary $S = \partial W$.

   **Answer:** $\frac{76\pi}{3}$

5. Find and prove a “product rule” expressing $\text{div}(f\mathbf{F})$ in terms of $\text{div}(\mathbf{F})$ and $\nabla f$.

   **Answer:** $\text{div}(f\mathbf{F}) = f\text{div}(\mathbf{F}) + \nabla f \cdot \mathbf{F}$. 